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2003 J. Phys. A: Math. Gen. 36 8003

(http://iopscience.iop.org/0305-4470/36/29/307)

J. Phys. A: Math. Gen. **36** (2003) 8003–8023 PII: S0305-4470(03)62717-4

An integral equation arising in two group neutron transport theory

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Received 28 April 2003
Published 8 July 2003
Online at stacks.iop.org/JPhysA/36/8003

Abstract
An integral equation describing the fuel distribution necessary to maintain a flat flux in a nuclear reactor in two group transport theory is reduced to the solution of a singular integral equation. The formalism developed enables the physical aspects of the problem to be better understood and its relationship with the corresponding diffusion theory model is highlighted. The integral equation is solved by reducing it to a non-singular Fredholm equation which is then evaluated numerically.

1. Introduction

In two earlier papers (Cassell and Williams 2003, Williams 2003), henceforth denoted by I and II, respectively, the problem of the conditions required to maintain a flat thermal flux in a nuclear reactor has been examined in the context of transport theory. In paper I, one-speed transport theory was employed and no other approximations made. It was then possible to obtain an exact solution of the singular integral equation that arose. In paper II, the two group model was introduced thereby enabling fission to be dealt with in a realistic manner for thermal reactors. It was necessary in II to introduce an approximation regarding the angular flux of thermal neutrons in the core which, however, was shown to lead to very small errors for weakly absorbing moderators. The outcome was an integral equation for the fuel distribution across the core necessary to give a flat flux and in addition the minimum amount of fuel for criticality. This integral equation takes the form

$$1 = \int_V \mathrm{d}\mathbf{r}' M(\mathbf{r}')[\eta \Sigma_{\mathrm{rm}} \Sigma_{\mathrm{atm}} P_{\mathrm{tf}}(|\mathbf{r}-\mathbf{r}'|) - \Sigma_{\mathrm{atm}} P_{\mathrm{t}}(|\mathbf{r}-\mathbf{r}'|)] \tag{1}$$

where $M(\mathbf{r}) = N(\mathbf{r})\sigma_{\mathrm{act}}/\Sigma_{\mathrm{atm}}$, $N(\mathbf{r})$ being the number density of fissile material at position $\mathbf{r}$. Also $\eta = \bar{\nu}\sigma_{\mathrm{ft}}/\sigma_{\mathrm{atc}}$, where σ_{ft} is the thermal microscopic fission cross section, σ_{act} is the thermal microscopic absorption cross section of the fuel and $\bar{\nu}$ is the mean number of neutrons per fission. Σ_{atm} is the macroscopic thermal absorption cross section of the moderator and Σ_{rm} is the slowing down cross section in the moderator. The kernels P_{tf} and P_{t} will be defined below. Equation (1) embodies all of the restrictions of the original classic problem defined by Goertzel (1956), who used diffusion theory; namely that the presence of the fuel shall not affect the diffusion or slowing down properties of the core region.

In II, equation (1) was solved for a slab system by a numerical method. The results obtained were sufficiently accurate to draw some important conclusions about the nature of the solution. In particular, it was shown that, for a system whose critical size was less than that required for minimum critical mass, the fuel distribution increased rapidly near the core-reflector interface. Such behaviour had been observed in diffusion theory but the only way to represent it mathematically was by the physically questionable addition of surface delta functions. In paper II, we were able to show that such an increase near the core edge was a natural consequence of transport theory and no supplementary delta functions are needed. Such a rapidly varying form of $M(\mathbf{r})$ near the surface placed a great strain upon the numerical method employed to solve equation (1), and it was necessary to employ over 1000 mesh points to obtain acceptable accuracy for such quantities as the total fuel mass. For this reason it seemed important to solve the equation by a method analogous to that used in I. This paper presents the mathematical procedure which leads to a solution and we are able to describe the general form of the mass distribution function over the bulk of the core and near the edge. Some numerical results are given in order to assess the accuracy of the earlier approximate methods and to illustrate the solution.

2. Derivation of the integral equation

Although the derivation of equation (1) is given in II, it is convenient to repeat that derivation here in a more transparent manner. Let us therefore write down the two group transport equations in one-dimensional geometry (Davison 1957)

$$\mu\frac{\partial\psi_{\mathrm{t}}(x,\mu)}{\partial x}+(\Sigma_{\mathrm{tm}}(x)+\Sigma_{\mathrm{tc}}(x))\psi_{\mathrm{t}}(x,\mu)=\frac{1}{2}(\Sigma_{\mathrm{stm}}(x)+\Sigma_{\mathrm{stc}}(x))\psi_{0\mathrm{t}}(x)+\frac{1}{2}\Sigma_{\mathrm{rm}}\psi_{0\mathrm{f}}(x) \quad (2)$$

$$\mu\frac{\partial\psi_{\mathrm{f}}(x,\mu)}{\partial x}+\Sigma_{\mathrm{fm}}(x)\psi_{\mathrm{f}}(x,\mu)=\frac{1}{2}(\Sigma_{\mathrm{sfm}}(x)-\Sigma_{\mathrm{rm}}(x))\psi_{0\mathrm{f}}(x)+\frac{1}{2}\bar{\nu}\Sigma_{\mathrm{f}t}(x)\psi_{0\mathrm{t}}(x). \quad (3)$$

In these equations, ψ_t and ψ_f are the thermal and fast angular fluxes, respectively. The cross sections have their usual meanings and are, in general, position dependent.

If now we introduce Goertzel's approximation that the presence of fuel does not influence slowing down, and in addition we stipulate that there is no absorption in the fast group, we have

$$\mu\frac{\partial\psi_{\mathrm{t}}(x,\mu)}{\partial x}+\Sigma_{\mathrm{tm}}\psi_{\mathrm{t}}(x,\mu)=\frac{1}{2}(\Sigma_{\mathrm{stm}}+\sigma_{\mathrm{stc}}N(x))\psi_{0\mathrm{t}}(x)+\frac{1}{2}\Sigma_{\mathrm{rm}}\psi_{0\mathrm{f}}(x)-\sigma_{\mathrm{tc}}N(x)\psi_{\mathrm{t}}(x,\mu) \quad (4)$$

$$\mu\frac{\partial\psi_{\mathrm{f}}(x,\mu)}{\partial x}+\Sigma_{\mathrm{fm}}\psi_{\mathrm{f}}(x,\mu)=\frac{1}{2}(\Sigma_{\mathrm{sfm}}-\Sigma_{\mathrm{rm}})\psi_{0\mathrm{f}}(x)+\frac{1}{2}\bar{\nu}\sigma_{\mathrm{ft}}N(x)\psi_{0\mathrm{t}}(x). \quad (5)$$

Assuming that the reflector extends to infinity, we simplify equation (5) by taking Fourier transforms in x, i.e. define

$$\bar{\psi}_{\mathrm{f}}(k,\mu)=\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx}\psi_{\mathrm{f}}(x,\mu) \quad (6)$$

whence equation (5) becomes for the scalar fast flux

$$\bar{\psi}_{0\mathrm{f}}(k) = \frac{\frac{1}{k}\tan^{-1}(k/\Sigma_{\mathrm{sfm}})}{1 - \frac{(\Sigma_{\mathrm{sfm}}-\Sigma_{\mathrm{rm}})}{k}\tan^{-1}(k/\Sigma_{\mathrm{sfm}})}\bar{\nu}\sigma_{\mathrm{ft}}\int_{-a}^{a}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx}N(x)\psi_{0\mathrm{t}}(x). \tag{7}$$

Inverting the transform leads to the following form:

$$\psi_{0\mathrm{f}}(x) = \bar{\nu}\sigma_{\mathrm{ft}}\int_{-a}^{\mathrm{a}}\mathrm{d}x'N(x')\psi_{0\mathrm{t}}(x')P_{\mathrm{f}}(|x-x'|) \tag{8}$$

where

$$P_{\mathrm{f}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kx}\frac{\frac{1}{k}\tan^{-1}(k/\Sigma_{\mathrm{sfm}})}{1 - \frac{(\Sigma_{\mathrm{sfm}}-\Sigma_{\mathrm{rm}})}{k}\tan^{-1}(k/\Sigma_{\mathrm{sfm}})}. \tag{9}$$

This Fourier integral is readily evaluated (Davison 1957) in the form

$$P_{\mathrm{f}}(x) = A_{\mathrm{f}}\,\mathrm{e}^{-\xi x} + \frac{1}{2}\int_{0}^{1}\frac{\mathrm{d}\mu}{\mu}g(c_{\mathrm{rm}},\mu)\,\mathrm{e}^{-\Sigma_{\mathrm{sfm}}x/\mu} \tag{10}$$

where

$$A_{\mathrm{f}} = \frac{\xi\left(\Sigma_{\mathrm{sfm}}^2 - \xi^2\right)}{c_{\mathrm{rm}}\left(\xi^2 - (1-c_{\mathrm{rm}})\Sigma_{\mathrm{sfm}}^2\right)\Sigma_{\mathrm{sfm}}} \tag{11}$$

$$\frac{1}{g(c,\mu)}\left(1 - \frac{c\mu}{2}\log\left(\frac{1+\mu}{1-\mu}\right)\right) + \left(\frac{c\pi\mu}{2}\right)^2 \tag{12}$$

and ξ is the root of

$$1 = \frac{c_{\mathrm{rm}}\Sigma_{\mathrm{sfm}}}{2\xi}\log\left(\frac{\Sigma_{\mathrm{sfm}}+\xi}{\Sigma_{\mathrm{sfm}}-\xi}\right) \tag{13}$$

where $c_{\mathrm{rm}} = (\Sigma_{\mathrm{sfm}} - \Sigma_{\mathrm{rm}})/\Sigma_{\mathrm{sfm}}$.

Equation (8) is a link between the fast and thermal fluxes throughout the core and reflector.

Now we must consider the thermal flux equation (4). By definition, the scalar flux in the core is a constant. However, as we have seen in both I and II, this does not imply that the associated angular flux $\psi_t(x,\mu)$ is spatially constant or isotropic. Nevertheless, we have shown in I and II that if the moderator is weakly absorbing, i.e. $\Sigma_{\mathrm{atm}}/\Sigma_{\mathrm{tm}} \ll 1$, then to a very good approximation

$$\psi_t(x,\mu) \simeq \tfrac{1}{2}\Psi \tag{14}$$

where Ψ is the constant scalar flux in the core. We shall use assumption (14) throughout. In view of (14) we can write for the core region

$$\Sigma_{\mathrm{tm}}\Psi = (\Sigma_{\mathrm{stm}} + \sigma_{\mathrm{stc}}N(x))\Psi + \Sigma_{\mathrm{rm}}\psi_{0\mathrm{f}}(x) - \sigma_{\mathrm{tc}}N(x)\Psi \tag{15}$$

which simplifies to

$$(\Sigma_{\mathrm{atm}} + \sigma_{\mathrm{atc}}N(x))\Psi = \Sigma_{\mathrm{rm}}\psi_{0\mathrm{f}}(x). \tag{16}$$

This equation relates the fuel density $N(x)$ to the fast scalar flux $\psi_{0\mathrm{f}}(x)$. Thus using (16) in (8), we find

$$1 + M(x) = \eta\int_{-a}^{a}\mathrm{d}x'M(x')\Sigma_{\mathrm{rm}}P_{\mathrm{f}}(|x-x'|). \tag{17}$$

Equation (17) is an integral equation for $M(x) = \sigma_{\mathrm{act}}N(x)/\Sigma_{\mathrm{atm}}$ and it appears therefore that the problem is solved once (17) is solved. Unfortunately, this is not so because (17) provides too drastic a simplification of the problem; the absence of any thermal diffusion kernel in (17) alerts us to this omission.

If we take a Fourier transform of equation (4), then it is easily seen that

$$\bar{\psi}_{0\mathrm{t}}(k) = \frac{\frac{1}{k}\tan^{-1}(k/\Sigma_{\mathrm{tm}})}{1-\frac{\Sigma_{\mathrm{stm}}}{k}\tan^{-1}(k/\Sigma_{\mathrm{tm}})}[\Sigma_{\mathrm{rm}}\bar{\psi}_{0\mathrm{f}}(k)+\sigma_{\mathrm{stc}}\langle N(x)\psi_{0\mathrm{t}}(x)\rangle]$$
$$-\frac{\sigma_{\mathrm{tc}}}{1-\frac{\Sigma_{\mathrm{stm}}}{k}\tan^{-1}(k/\Sigma_{\mathrm{tm}})}\int_{-1}^{1}\frac{\mathrm{d}\mu}{\Sigma_{\mathrm{tm}}-\mathrm{i}k\mu}\langle N(x)\psi_{\mathrm{t}}(x,\mu)\rangle \tag{18}$$

where the angular brackets define a Fourier transform of the quantity in those brackets.

If we now use the assumption that the angular thermal flux is isotropic and spatially constant in the core (but not of course in the reflector) then equation (18) becomes

$$\bar{\psi}_{0\mathrm{t}}(k) = \frac{\frac{1}{k}\tan^{-1}(k/\Sigma_{\mathrm{tm}})}{1-\frac{\Sigma_{\mathrm{stm}}}{k}\tan^{-1}(k/\Sigma_{\mathrm{tm}})}[\Sigma_{\mathrm{rm}}\bar{\psi}_{0\mathrm{f}}(k)-\sigma_{\mathrm{atc}}\langle N(x)\Psi\rangle]. \tag{19}$$

Inverting equation (19) we find

$$\psi_{0\mathrm{t}}(x) = \Sigma_{\mathrm{rm}}\int_{-\infty}^{\infty}\mathrm{d}x'\psi_{0\mathrm{f}}(x')P_{\mathrm{t}}(|x-x'|)-\sigma_{\mathrm{atc}}\Psi\int_{-a}^{a}\mathrm{d}x'N(x')P_{\mathrm{t}}(|x-x'|) \tag{20}$$

with

$$P_{\mathrm{t}}(x) = A_{\mathrm{t}}\,\mathrm{e}^{-\nu x}+\frac{1}{2}\int_{0}^{1}\frac{\mathrm{d}\mu}{\mu}g(c_{\mathrm{sm}},\mu)\,\mathrm{e}^{-\Sigma_{\mathrm{tm}}x/\mu} \tag{21}$$

where

$$A_{\mathrm{t}} = \frac{\nu\left(\Sigma_{\mathrm{tm}}^{2}-\nu^{2}\right)}{c_{\mathrm{sm}}\left(\nu^{2}-(1-c_{\mathrm{sm}})\Sigma_{\mathrm{tm}}^{2}\right)\Sigma_{\mathrm{tm}}} \tag{22}$$

ν is the root of

$$1 = \frac{c_{\mathrm{sm}}\Sigma_{\mathrm{tm}}}{2\nu}\log\left(\frac{\Sigma_{\mathrm{tm}}+\nu}{\Sigma_{\mathrm{tm}}-\nu}\right) \tag{23}$$

with $c_{\mathrm{sm}} = \Sigma_{\mathrm{stm}}/\Sigma_{\mathrm{tm}}$.

If we now replace $\psi_{0\mathrm{f}}(x)$ in equation (20) by equation (8), we find

$$\psi_{0\mathrm{t}}(x) = \Psi\int_{-a}^{a}\mathrm{d}x'M(x')[\eta\Sigma_{\mathrm{rm}}\Sigma_{\mathrm{atm}}P_{\mathrm{tf}}(|x-x'|)-\Sigma_{\mathrm{atm}}P_{\mathrm{t}}(|x-x'|)] \tag{24}$$

where

$$P_{\mathrm{tf}}(|x-x'|) = \int_{-\infty}^{\infty}\mathrm{d}x''P_{\mathrm{t}}(|x-x''|)P_{\mathrm{f}}(|x''-x'|). \tag{25}$$

Equation (24) is valid for all values of x but if we restrict x to the core region only where $\psi_{0\mathrm{t}} = \Psi$, then it reduces to

$$1 = \int_{-a}^{a}\mathrm{d}x'M(x')[\eta\Sigma_{\mathrm{rm}}\Sigma_{\mathrm{atm}}P_{\mathrm{tf}}(|x-x'|)-\Sigma_{\mathrm{atm}}P_{\mathrm{t}}(|x-x'|)] \tag{26}$$

which is another integral equation for $M(x)$ and is the one-dimensional analogue of equation (1). Equation (26) is to be used rather than equation (17) because it contains information about thermal neutron diffusion in the system via $P_{\mathrm{t}}(x)$.

Physically, we can define $\Sigma_{\mathrm{rm}}P_{\mathrm{f}}(x)$ as the slowing down kernel and $\Sigma_{\mathrm{atm}}P_{\mathrm{t}}(x)$ as the diffusion kernel. The kernel $\Sigma_{\mathrm{rm}}\Sigma_{\mathrm{atm}}P_{\mathrm{tf}}(x)$ denotes the combined effect of slowing down and diffusion. The algebraic form of $P_{\mathrm{tf}}(x)$ has been determined in II and can be written as

$$P_{\mathrm{tf}}(x) = \frac{A_{\nu}}{2\nu}\,\mathrm{e}^{-\nu x}+\frac{A_{\xi}}{2\xi}\,\mathrm{e}^{-\xi x}+\frac{1}{4\Sigma_{\mathrm{tm}}}\int_{0}^{1}\mathrm{d}\mu B_{0}(\mu)\,\mathrm{e}^{-\Sigma_{\mathrm{tm}}x/\mu}+\frac{1}{4\Sigma_{\mathrm{tm}}}\int_{1}^{1/\hat{r}}\mathrm{d}\mu B_{1}(\mu)\,\mathrm{e}^{-\Sigma_{\mathrm{tm}}x/\mu} \tag{27}$$

where

$$A_\nu = \frac{\log\left(\frac{\Sigma_{\rm sfm}+\nu}{\Sigma_{\rm sfm}-\nu}\right)}{\left[1-\frac{c_{\rm rm}\Sigma_{\rm sfm}}{2\nu}\log\left(\frac{\Sigma_{\rm sfm}+\nu}{\Sigma_{\rm sfm}-\nu}\right)\right]}\frac{\nu\left(\Sigma_{\rm tm}^2-\nu^2\right)}{c_{\rm sm}\Sigma_{\rm tm}\left[\nu^2-\Sigma_{\rm tm}^2(1-c_{\rm sm})\right]} \tag{28}$$

$$A_\xi = \frac{\log\left(\frac{\Sigma_{\rm tm}+\xi}{\Sigma_{\rm tm}-\xi}\right)}{\left[1-\frac{c_{\rm sm}\Sigma_{\rm tm}}{2\xi}\log\left(\frac{\Sigma_{\rm tm}+\xi}{\Sigma_{\rm tm}-\xi}\right)\right]}\frac{\xi\left(\Sigma_{\rm sfm}^2-\xi^2\right)}{c_{\rm rm}\Sigma_{\rm sfm}\left[\xi^2-\Sigma_{\rm sfm}^2(1-c_{\rm rm})\right]} \tag{29}$$

$$B_0(\mu) = \left\{\begin{matrix}\log\left(\frac{1+\hat r\mu}{1-\hat r\mu}\right)+\log\left(\frac{1+\mu}{1-\mu}\right)\\ -\frac{1}{2}c_{\rm rm}\hat r\mu\left(\pi^2+\log^2\left(\frac{1+\hat r\mu}{1-\hat r\mu}\right)\right)-\frac{1}{2}c_{\rm sm}\mu\left(\pi^2+\log^2\left(\frac{1+\mu}{1-\mu}\right)\right)\end{matrix}\right\} g(c_{\rm sm},\mu)g(c_{\rm rm},\hat r\mu) \tag{30}$$

$$B_1(\mu) = \frac{\log\left(\frac{\mu+1}{\mu-1}\right)}{1-\frac{1}{2}c_{\rm sm}\mu\log\left(\frac{\mu+1}{\mu-1}\right)}g(c_{\rm rm},\hat r\mu). \tag{31}$$

Now it is convenient to measure distance in units of the thermal mean free path $1/\Sigma_{\rm tm}$. Thus we write $\bar x = x\Sigma_{\rm tm}$, $\bar\xi = \xi/\Sigma_{\rm tm}$, $\bar\nu = \nu/\Sigma_{\rm tm}$ and $\hat r = \Sigma_{\rm sfm}/\Sigma_{\rm tm}$. With these scalings and omitting the overbar, equation (26) becomes

$$\frac{1}{1-c_{\rm sm}} = \int_{-a}^{a}{\rm d}x' M(x')[\eta(1-c_{\rm rm})\hat r P_{\rm tf}(|x-x'|)-P_{\rm t}(|x-x'|)] \tag{32}$$

where $P_{\rm tf}$ is written in an abbreviated notation as

$$P_{\rm tf}(x) = C_{\rm t}\,{\rm e}^{-\nu x}+C_{\rm f}\,{\rm e}^{-\xi x}+\int_0^{1/\hat r}{\rm d}\mu E(\mu)\,{\rm e}^{-x/\mu} \tag{33}$$

with

$$C_{\rm t} = \frac{(1-\nu^2)\log\left(\frac{\hat r+\nu}{\hat r-\nu}\right)}{2c_{\rm sm}(\nu^2-1+c_{\rm sm})\left[1-\frac{c_{\rm rm}\hat r}{2\nu}\log\left(\frac{\hat r+\nu}{\hat r-\nu}\right)\right]} \tag{34}$$

$$C_{\rm f} = \frac{(\hat r^2-\xi^2)\log\left(\frac{1+\xi}{1-\xi}\right)}{2c_{\rm rm}\hat r(\xi^2-\hat r^2(1-c_{\rm rm}))\left[1-\frac{c_{\rm sm}}{2\xi}\log\left(\frac{1+\xi}{1-\xi}\right)\right]} \tag{35}$$

and

$$\begin{aligned} E(\mu) &= \tfrac{1}{4}B_0(\mu) \qquad 0\leqslant\mu\leqslant 1\\ &= \tfrac{1}{4}B_1(\mu) \qquad 1\leqslant\mu\leqslant 1/\hat r.\end{aligned}$$

The roots ν and ξ are found from

$$1 = \frac{c_{\rm sm}}{2\nu}\log\left(\frac{1+\nu}{1-\nu}\right) \qquad 1 = \frac{\hat r c_{\rm rm}}{2\xi}\log\left(\frac{\hat r+\xi}{\hat r-\xi}\right).$$

In the next section we will describe a method for solving equation (32).

3. The integral equation

3.1. Structure of the solution

Before discussing the method for solving equation (32), it is useful to recapitulate some results from I and II. In I, which was for one-speed transport theory, we found that the solution for $c(\tau)$ [the quantity analogous to $M(x)$], takes the form

$$c(\tau) = 1+\int_0^1{\rm d}\mu B(\mu)\cosh(\tau/\mu). \tag{36}$$

The integral term in (36) is important only near the core-reflector interface, where it diverges to infinity as $1/\sqrt{a-\tau}$. On the other hand, we also know from Goertzel (1956) and from II, that in diffusion theory for the two group case

$$M(x) = \frac{1}{\eta - 1} + G_0 \cos \lambda x + \Lambda[\delta(x-a) + \delta(x+a)] \tag{37}$$

where $\lambda = \sqrt{\eta - 1}/L_s$, L_s being the slowing down length. These two solutions together with (33), suggest that in two group transport theory the solution will assume the form

$$M(x) = A_0 + G_0 \cos \lambda x + \int_0^{1/\hat{r}} \mathrm{d}\mu A(\mu) \cosh(x/\mu). \tag{38}$$

We will show that equation (38) satisfies equation (32) provided A_0, G_0, λ and $A(\mu)$ obey certain conditions. It is clear that, in transport theory, the physically questionable delta functions are replaced by an ordinary function via the integral term in equation (38).

3.2. A subsidiary integral equation

Whilst we noted that equation (17) was not the complete solution because it ignores the influence of thermal neutrons in the reflector, it does have some interest. This can best be shown by replacing $\Sigma_{\mathrm{rm}} P_{\mathrm{f}}(x)$ by its diffusion theory counterpart, namely:

$$\Sigma_{\mathrm{rm}} P_{\mathrm{f}}(x) \simeq \frac{1}{2L_s} \mathrm{e}^{-x/L_s} \equiv G_{\mathrm{f}}(x). \tag{39}$$

Thus equation (17) becomes

$$1 + M(x) = \eta \int_{-a}^{a} \mathrm{d}x' M(x') G_{\mathrm{f}}(|x - x'|). \tag{40}$$

But we can readily convert this integral equation into the following differential form:

$$L_s^2 M''(x) + (\eta - 1) M(x) = 1. \tag{41}$$

Equation (33) is simply the equation derived by Goertzel (1956) in the two group model. Thus equation (17), whilst not complete, does contain important information. The general solution of (41) with $\lambda = \sqrt{\eta - 1}/L_s$ is

$$M(x) = \frac{1}{\eta - 1} + G_0 \cos \lambda x. \tag{42}$$

However, G_0 is an unknown constant and to obtain this it is necessary for equation (42) to also satisfy the diffusion theory counterpart of equation (26) which is

$$1 = \int_{-a}^{a} \mathrm{d}x' M(x') \left[\frac{1}{2} \left(\frac{\eta L}{L^2 - L_s^2} - \frac{1}{L} \right) \mathrm{e}^{-|x-x'|/L} - \frac{\eta L_s}{2(L^2 - L_s^2)} \mathrm{e}^{-|x-x'|/L_s} \right] \tag{43}$$

where we have noted that in diffusion theory

$$\Sigma_{\mathrm{atm}} P_{\mathrm{t}}(x) \simeq \frac{1}{2L} \mathrm{e}^{-x/L} \equiv G_{\mathrm{t}}(x) \tag{44}$$

L being the diffusion length.

G_0 can be determined by ensuring that equation (42) is consistent with equation (43). This is discussed in II. Indeed, it can also be shown that an equation of the form of (37) containing surface delta functions satisfies (43) and leads to a value for Λ.

3.3. The general solution for $M(x)$

We seek a solution to equation (32) in the form defined by equation (38). If this is done and the integrals over x' carried out, then it is found that equation (32) takes the form

$$\frac{1}{1-c_{\mathrm{sm}}} = Z_1 A_0 + Z_2 \cos\lambda x + Z_3 \cosh\nu x + Z_4 \cosh\xi x + \int_0^{1/\hat{r}} \mathrm{d}\mu\, Z_5(\mu)\cosh(x/\mu). \tag{45}$$

For this to be satisfied, it is necessary that

$$Z_1 A_0 = \frac{1}{1-c_{\mathrm{sm}}}$$

$Z_2 = 0,\ Z_3 = 0,\ Z_4 = 0$ and also $Z_5(\mu) = 0$. These relations enable us to find equations for $A_0,\ G_0\ \lambda$ and $A(\mu)$.

In order to calculate A_0 we use the relation (A6) in the appendix with $\lambda = 0$ to get

$$\int_0^1 \mathrm{d}\mu g(c_{\mathrm{sm}},\mu) + \frac{2A_{\mathrm{t}}}{\nu} = \frac{1}{1-c_{\mathrm{sm}}} \tag{46}$$

which leads to $A_0 = 1/(\eta - 1)$.

Setting $Z_2 = 0$, we find the following equation:

$$0 = \frac{2\nu}{\nu^2+\lambda^2}[\eta\hat{r}(1-c_{\mathrm{rm}})C_{\mathrm{t}} - A_{\mathrm{t}}] + \frac{2\xi}{\xi^2+\lambda^2}\eta\hat{r}(1-c_{\mathrm{rm}})C_{\mathrm{f}} + \frac{1}{2}\eta\hat{r}(1-c_{\mathrm{rm}}) \times\left[\int_0^1 \frac{\mathrm{d}\mu\mu B_0(\mu)}{1+\lambda^2\mu^2} + \int_1^{1/\hat{r}} \frac{\mathrm{d}\mu\mu B_1(\mu)}{1+\lambda^2\mu^2}\right] - \int_0^1 \frac{\mathrm{d}\mu g(c_{\mathrm{sm}},\mu)}{1+\lambda^2\mu^2}. \tag{47}$$

We can then use the following relations (see the appendix)

$$\frac{2\nu A_{\mathrm{t}}}{\lambda^2+\nu^2} + \int_0^1 \frac{\mathrm{d}\mu g(c_{\mathrm{sm}},\mu)}{1+\lambda^2\mu^2} = \frac{\tan^{-1}\lambda}{\lambda - c_{\mathrm{sm}}\tan^{-1}\lambda} \tag{48}$$

and

$$\frac{2\nu C_{\mathrm{t}}}{\nu^2+\lambda^2} + \frac{2\xi C_{\mathrm{f}}}{\xi^2+\lambda^2} + \frac{1}{2}\int_0^1 \frac{\mathrm{d}\mu\mu B_0(\mu)}{1+\lambda^2\mu^2} + \frac{1}{2}\int_1^{1/\hat{r}} \frac{\mathrm{d}\mu\mu B_1(\mu)}{1+\lambda^2\mu^2} = \frac{\tan^{-1}(\lambda/\hat{r})}{\lambda - c_{\mathrm{rm}}\hat{r}\tan^{-1}(\lambda/\hat{r})}\frac{\tan^{-1}(\lambda)}{\lambda - c_{\mathrm{sm}}\tan^{-1}(\lambda)} \tag{49}$$

to reduce equation (47) to

$$1 = [\eta(1-c_{\mathrm{rm}}) + c_{\mathrm{rm}}]\frac{\tan^{-1}(\lambda/\hat{r})}{\lambda/\hat{r}}. \tag{50}$$

Equation (50) gives the root λ. It is interesting to note that if $1-c_{\mathrm{rm}} \ll 1$, this root approximates to

$$\lambda \simeq \frac{\sqrt{\eta-1}}{L_{\mathrm{s}}} \tag{51}$$

where $L_{\mathrm{s}} = 1/\hat{r}\sqrt{3(1-c_{\mathrm{rm}})}$ in agreement with diffusion theory. Equation (50) gives the generalized value of λ for the transport equation.

The equations $Z_3 = 0$ and $Z_4 = 0$ give the following conditions:

$$\frac{1}{\eta-1} + \frac{G_0\nu}{\nu^2+\lambda^2}(\nu\cos\lambda a - \lambda\sin\lambda a) = \frac{\nu}{2}\int_0^{1/\hat{r}} \mathrm{d}\mu\,\psi(\mu)\left[\frac{1}{1-\nu\mu} - \frac{\mathrm{e}^{-2a/\mu}}{1+\nu\mu}\right] \tag{52}$$

$$\frac{1}{\eta-1} + \frac{G_0\xi}{\xi^2+\lambda^2}(\xi\cos\lambda a - \lambda\sin\lambda a) = \frac{\xi}{2}\int_0^{1/\hat{r}} \mathrm{d}\mu\,\psi(\mu)\left[\frac{1}{1-\xi\mu} - \frac{\mathrm{e}^{-2a/\mu}}{1+\xi\mu}\right] \tag{53}$$

where

$$\psi(\mu) = \mu A(\mu)\,\mathrm{e}^{a/\mu}. \tag{54}$$

Finally, with $Z_5(\mu) = 0$, we find the integral equation

$$0 = \frac{2}{\eta - 1} + \frac{2G_0}{1 + \lambda^2\mu^2}(\cos\lambda a - \lambda\mu\sin\lambda a) + Q(\mu)\psi(\mu) + \int_0^{1/\hat{r}} \mathrm{d}\mu_0 \psi(\mu_0)\left[\frac{1}{\mu_0 - \mu} + \frac{\mathrm{e}^{-2a/\mu_0}}{\mu_0 + \mu}\right]. \tag{55}$$

Equation (55) is a singular integral equation for $\psi(\mu)$ with $0 \leqslant \mu \leqslant 1/\hat{r}$. The coefficient $Q(\mu)$ is defined as $G(\mu)/F(\mu)$ where

$$\begin{aligned}
G(\mu) &= \eta\hat{r}(1 - c_{\mathrm{rm}})\left\{\frac{2\nu C_{\mathrm{t}}}{1 - \nu^2\mu^2} + \frac{2\xi C_{\mathrm{f}}}{1 - \xi^2\mu^2} + \frac{1}{2}P.\int_0^1 \frac{\mathrm{d}\mu'\mu' B_0(\mu')}{\mu'^2 - \mu^2} + \frac{1}{2}\int_1^{1/\hat{r}} \frac{\mathrm{d}\mu'\mu' B_1(\mu')}{\mu'^2 - \mu^2}\right\} \\
&\quad - \left\{\frac{2\nu A_t}{1 - \nu^2\mu^2} + P.\int_0^1 \frac{\mathrm{d}\mu' g(c_{\mathrm{sm}}, \mu')}{\mu'^2 - \mu^2}\right\} \qquad 0 \leqslant \mu \leqslant 1 \\
&= \eta\hat{r}(1 - c_{\mathrm{rm}})\left\{\frac{2\nu C_t}{1 - \nu^2\mu^2} + \frac{2\xi C_f}{1 - \xi^2\mu^2} + \frac{1}{2}\int_0^1 \frac{\mathrm{d}\mu'\mu' B_0(\mu')}{\mu'^2 - \mu^2}\right. \\
&\quad \left. + \frac{1}{2}P.\int_1^{1/\hat{r}} \frac{\mathrm{d}\mu'\mu' B_1(\mu')}{\mu'^2 - \mu^2}\right\} - \left\{\frac{2\nu A_{\mathrm{t}}}{1 - \nu^2\mu^2} + \int_0^1 \frac{\mathrm{d}\mu' g(c_{\mathrm{sm}}, \mu')}{\mu'^2 - \mu^2}\right\} \\
&\quad 1 \leqslant \mu \leqslant 1/\hat{r}
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
F(\mu) &= \eta\hat{r}(1 - c_{\mathrm{rm}})\frac{1}{4}B_0(\mu) - \frac{1}{2\mu}g(c_{\mathrm{sm}}, \mu) \qquad 0 \leqslant \mu \leqslant 1 \\
&= \eta\hat{r}(1 - c_{\mathrm{rm}})\frac{1}{4}B_1(\mu) \qquad 1 \leqslant \mu \leqslant 1/\hat{r}.
\end{aligned} \tag{57}$$

Now it is shown in I that

$$\begin{aligned}
&\frac{2\nu A_{\mathrm{t}}}{1 - \nu^2\mu^2} + \int_0^1 \frac{\mathrm{d}\mu' g(c_{\mathrm{sm}}, \mu')}{\mu'^2 - \mu^2} \\
&\quad = \frac{1}{2\mu}g(c_{\mathrm{sm}}, \mu)\left\{-L(\mu) + \frac{1}{2}c_{\mathrm{sm}}\mu(\pi^2 + L^2(\mu))\right\} \qquad 0 < \mu < 1 \\
&\quad = -\frac{1}{2\mu}\frac{\bar{L}(\mu)}{1 - \frac{1}{2}\mu c_{\mathrm{sm}}\bar{L}(\mu)} \qquad 1 < \mu < 1/\hat{r}
\end{aligned} \tag{58}$$

where $L(\mu) = \log((1 + \mu)/(1 - \mu))$ and $\bar{L}(\mu) = \log((\mu + 1)/(\mu - 1))$. We also show in the appendix that

$$\begin{aligned}
&\frac{2\nu C_{\mathrm{t}}}{1 - \nu^2\mu^2} + \frac{2\xi C_{\mathrm{f}}}{1 - \xi^2\mu^2} + \frac{1}{2}P.\int_0^1 \frac{\mathrm{d}\mu'\mu' B_0(\mu')}{\mu'^2 - \mu^2} + \frac{1}{2}\int_1^{1/\hat{r}} \frac{\mathrm{d}\mu'\mu' B_1(\mu')}{\mu'^2 - \mu^2} \\
&\quad = \frac{1}{4}g(c_{\mathrm{sm}}, \mu)g(c_{\mathrm{rm}}, \hat{r}\mu)\left[\pi^2 - \left(L(\mu) - \frac{1}{2}c_{\mathrm{sm}}\mu(\pi^2 + L^2(\mu))\right)\right. \\
&\quad\quad \left.\times\left(L(\hat{r}\mu) - \frac{1}{2}c_{\mathrm{rm}}\hat{r}\mu(\pi^2 + L^2(\hat{r}\mu))\right)\right] \qquad 0 \leqslant \mu \leqslant 1
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
&\frac{2\nu C_{\mathrm{t}}}{1 - \nu^2\mu^2} + \frac{2\xi C_{\mathrm{f}}}{1 - \xi^2\mu^2} + \frac{1}{2}\int_0^1 \frac{\mathrm{d}\mu'\mu' B_0(\mu')}{\mu'^2 - \mu^2} + \frac{1}{2}P.\int_1^{1/\hat{r}} \frac{\mathrm{d}\mu'\mu' B_1(\mu')}{\mu'^2 - \mu^2} \\
&\quad = -\frac{1}{4}B_1(\mu)\left(L(\hat{r}\mu) - \frac{1}{2}c_{\mathrm{rm}}\hat{r}\mu(\pi^2 + L^2(\hat{r}\mu))\right) \qquad 1 \leqslant \mu \leqslant 1/\hat{r}.
\end{aligned} \tag{60}$$

Equations (59) and (60) can also be obtained from equation (49) by allowing $\lambda \to \frac{i}{\mu} \pm 0$. Then if we define $p = \eta\hat{r}(1 - c_{\mathrm{rm}})$, and

$$S_{\mathrm{t}}(\mu) = L(\mu) - \tfrac{1}{2}c_{\mathrm{sm}}\mu(\pi^2 + L^2(\mu)) \tag{61}$$

$$S_{\mathrm{f}}(\mu) = L(\hat{r}\mu) - \tfrac{1}{2}c_{\mathrm{rm}}\hat{r}\mu(\pi^2 + L^2(\hat{r}\mu)) \tag{62}$$

we have

$$\begin{aligned} Q(\mu) &= \frac{p\mu g(c_{\mathrm{rm}}, \hat{r}\mu)[\pi^2 - S_{\mathrm{t}}(\mu)S_{\mathrm{f}}(\mu)] + 2S_{\mathrm{t}}(\mu)}{p\mu g(c_{\mathrm{rm}}, \hat{r}\mu)[S_{\mathrm{t}}(\mu) + S_{\mathrm{f}}(\mu)] - 2} \qquad 0 \leqslant \mu \leqslant 1 \\ &= -S_{\mathrm{f}}(\mu) + \frac{2}{p\mu g(c_{\mathrm{rm}}, \hat{r}\mu)} \qquad 1 \leqslant \mu \leqslant 1/\hat{r}. \end{aligned} \tag{63}$$

We note that $Q(\mu)$ is continuous at $\mu = 1$, but the gradient is not.

In the special case when $\hat{r} = 1$, $Q(\mu)$ simplifies to

$$Q(\mu) = -L(\mu) + \frac{1}{2}\mu(\pi^2 + L^2(\mu))\frac{(\alpha + 1)c_{\mathrm{sm}}g(c_{\mathrm{sm}}, \mu) + \alpha c_{\mathrm{rm}}g(c_{\mathrm{rm}}, \mu)}{(\alpha + 1)g(c_{\mathrm{sm}}, \mu) + \alpha g(c_{\mathrm{rm}}, \mu)} \tag{64}$$

where $\alpha = \eta(1 - c_{\mathrm{rm}})/(c_{\mathrm{rm}} - c_{\mathrm{sm}})$.

Thus, we now have the necessary relations for calculating the mass distribution function $M(x)$. An important integral quantity which is proportional to the amount of fuel in the core is

$$M_{\mathrm{T}} = \int_{-a}^{a} \mathrm{d}x\, M(x) = \frac{2a}{\eta - 1} + \frac{2G_0 \sin \lambda a}{\lambda} + \int_0^{1/\hat{r}} \mathrm{d}\mu\, \psi(\mu)(1 - \mathrm{e}^{-2a/\mu}). \tag{65}$$

We note that G_0 is unknown but enters equations (52) and (53) and hence we have G_0 in terms of $\psi(\mu)$.

4. Solution of the integral equation

In order to solve equation (55) it is convenient to write it in the following form:

$$Q(\mu)\psi(\mu) + P\int_0^{1/\hat{r}} \frac{\mathrm{d}\mu_0 \psi(\mu_0)}{\mu_0 - \mu} = -f(\mu) \tag{66}$$

where

$$f(\mu) = \frac{2}{\eta - 1} + \frac{2G_0}{1 + \lambda^2\mu^2}(\cos \lambda a - \lambda\mu \sin \lambda a) + \int_0^{1/\hat{r}} \frac{\mathrm{d}\mu_0 \psi(\mu_0)}{\mu_0 + \mu}\, \mathrm{e}^{-2a/\mu_0}. \tag{67}$$

We now reduce equation (66) to a Hilbert problem by introducing the following complex function:

$$\Phi(z) = \int_0^{1/\hat{r}} \frac{\mathrm{d}\mu_0 \psi(\mu_0)}{\mu_0 - z}. \tag{68}$$

Equation (66) can then be written as

$$(Q(\mu) + \mathrm{i}\pi)\Phi^+(\mu) - (Q(\mu) - \mathrm{i}\pi)\Phi^-(\mu) = -2\pi\mathrm{i}\, f(\mu). \tag{69}$$

Let

$$\Omega(z) = \frac{1}{2\pi\mathrm{i}}\int_0^{1/\hat{r}} \log\left(\frac{Q(\mu) - \mathrm{i}\pi}{Q(\mu) + \mathrm{i}\pi}\right)\frac{\mathrm{d}\mu}{\mu - z} \tag{70}$$

or

$$\Omega(z) = -\frac{1}{\pi}\int_0^{1/\hat{r}} \frac{\Theta(\mu)\, \mathrm{d}\mu}{\mu - z} \tag{71}$$

where

$$\Theta(\mu) = \arg(Q(\mu) + \mathrm{i}\pi) = \tan^{-1}(\pi/Q(\mu)) \tag{72}$$

and $\Theta(\mu)$ goes from $\pi/2$ to 0 as μ goes from 0 to $1/\hat{r}$. Now as $\mu \to 0$, $Q(\mu) = O(\mu)$ so $\Theta(\mu) = \pi/2 + O(\mu)$. Hence $\Omega(z) - \frac{1}{2}\log(-z)$ has a finite limit as $z \to 0$. Also

$$\Theta(\mu) = O\left(\frac{1}{\log^2(1/\hat{r} - \mu)}\right) \tag{73}$$

as $\mu \to 1/\hat{r}$ and $\Omega(z)$ has a finite limit as $z \to 1/\hat{r}$. Now set

$$X(z) = \frac{1}{z}\,\mathrm{e}^{\Omega(z)}. \tag{74}$$

Then

$$X(z) = (-z)^{-1/2}(a_0 + o(1)) \tag{75}$$

as $z \to 0$ for some non-zero constant a_0. For the most general solution we aim to get

$$X(z) = (-z)^{\alpha}(a_0 + o(1)) \tag{76}$$

with $-1 < \alpha \leqslant 0$. Also $X(z)$ has a finite non-zero limit as $z \to 1/\hat{r}$. Further,

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{Q(\mu) - \mathrm{i}\pi}{Q(\mu) + \mathrm{i}\pi} \tag{77}$$

so equation (69) can be written as

$$\frac{\Phi^+(\mu)}{X^+(\mu)} - \frac{\Phi^-(\mu)}{X^-(\mu)} = -\frac{2\pi\mathrm{i}\, f(\mu)}{X^+(\mu)(Q(\mu) + \mathrm{i}\pi)}. \tag{78}$$

Now since $\Phi(z) = O(1/z)$ at infinity and $X(z) \sim 1/z$,

$$\frac{\Phi(z)}{X(z)} = K - \int_0^{1/\hat{r}} \frac{\mathrm{d}\mu\, f(\mu)}{X^+(\mu)(Q(\mu) + \mathrm{i}\pi)(\mu - z)} \tag{79}$$

for some constant K. On using

$$\psi(\mu) = \frac{1}{2\pi\mathrm{i}}(\Phi^+(\mu) - \Phi^-(\mu)) \tag{80}$$

and replacing $f(\mu)$ by (67), we obtain a non-singular integral equation for $\psi(\mu)$. The two constants K and G_0 can be determined by equations (52) and (53).

To proceed we write

$$\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)} = \frac{1}{X^+(\mu)}\left(1 - \frac{Q(\mu) - \mathrm{i}\pi}{Q(\mu) + \mathrm{i}\pi}\right) = \frac{2\pi\mathrm{i}}{X^+(\mu)(Q(\mu) + \mathrm{i}\pi)} \tag{81}$$

so that

$$\int_0^{1/\hat{r}} \frac{\mathrm{d}\mu}{X^+(\mu)(Q(\mu) + \mathrm{i}\pi)(\mu - z)} = \frac{1}{2\pi\mathrm{i}} \int^{(0-,\frac{1}{\hat{r}}-)} \frac{\mathrm{d}\zeta}{X(\zeta)(\zeta - z)}. \tag{82}$$

For large z, from (71) and (74)

$$\frac{1}{X(z)} = z - \vartheta_0 + O(1/z) \tag{83}$$

with

$$\vartheta_0 = \frac{1}{\pi}\int_0^{1/\hat{r}} \Theta(\mu)\,\mathrm{d}\mu \tag{84}$$

and so

$$\int_0^{1/\hat{r}} \frac{d\mu}{X^+(\mu)(Q(\mu)+i\pi)(\mu-z)} = \frac{1}{X(z)} - z + \vartheta_0. \tag{85}$$

Further,

$$\int_0^{1/\hat{r}} \frac{d\mu}{X^+(\mu)(Q(\mu)+i\pi)(\mu-z)(\mu-\zeta)} = \frac{1}{z-\zeta}\left(\frac{1}{X(z)} - \frac{1}{X(\zeta)}\right) - 1. \tag{86}$$

Using (85) and (86) to evaluate the integral in (79) we obtain after some manipulation,

$$\Phi(z) = (K + H(z) + G_0 h(z))X(z) - H^*(z) - G_0 h^*(z) \tag{87}$$

with

$$H(z) = \frac{2(z-\vartheta_0)}{\eta-1} + \int_0^{1/\hat{r}} d\mu\, e^{-2a/\mu}\psi(\mu)\left(\frac{1}{(z+\mu)X(-\mu)} + 1\right) \tag{88}$$

$$H^*(z) = \frac{2}{\eta-1} + \int_0^{1/\hat{r}} \frac{d\mu\,\psi(\mu)}{z+\mu}\, e^{-2a/\mu} \tag{89}$$

$$h(z) = \frac{e^{i\lambda a}}{(1-i\lambda z)X(-i/\lambda)} + \frac{e^{-i\lambda a}}{(1+i\lambda z)X(i/\lambda)} - \frac{2\sin\lambda a}{\lambda} \tag{90}$$

and

$$h^*(z) = \frac{e^{i\lambda a}}{1-i\lambda z} + \frac{e^{-i\lambda a}}{1+i\lambda z}. \tag{91}$$

Consider now the constraints of equations (52) and (53). The first of these can be written as

$$\frac{2}{\eta-1} + \frac{2G_0\nu}{\nu^2+\lambda^2}(\nu\cos\lambda a - \lambda\sin\lambda a) + \Phi(1/\nu) + \nu\int_0^{1/\hat{r}} \frac{d\mu\,\psi(\mu)}{1+\nu\mu}\, e^{-2a/\mu} = 0 \tag{92}$$

i.e.

$$H^*(1/\nu) + G_0 h^*(1/\nu) + \Phi(1/\nu) = 0. \tag{93}$$

Then from equation (87),

$$K + H(1/\nu) + G_0 h(1/\nu) = 0. \tag{94}$$

Likewise (53) gives

$$K + H(1/\xi) + G_0 h(1/\xi) = 0. \tag{95}$$

From equations (94) and (95), we find

$$K = \frac{1}{k(1/\nu, 1/\xi)}(h(1/\xi)H(1/\nu) - h(1/\nu)H(1/\xi)) \tag{96}$$

and

$$G_0 = \frac{1}{k(1/\nu, 1/\xi)}(H(1/\xi) - H(1/\nu)) \tag{97}$$

where we have put

$$k(z,\zeta) = h(z) - h(\zeta) \tag{98}$$

which becomes

$$k(z,\zeta) = i\lambda(z-\zeta)\left\{\frac{e^{i\lambda a}}{(1-i\lambda z)(1-i\lambda\zeta)X(-i/\lambda)} - \frac{e^{-i\lambda a}}{(1+i\lambda z)(1+i\lambda\zeta)X(i/\lambda)}\right\}. \tag{99}$$

From equations (80) and (87),

$$\psi(\mu) = \frac{1}{2\pi \mathrm{i}}(X^+(\mu) - X^-(\mu))(K + H(\mu) + G_0 h(\mu)) \tag{100}$$

and from (71), (72) and (74)

$$\frac{1}{2\pi \mathrm{i}}(X^+(\mu) - X^-(\mu)) = -\frac{\exp\left(-\frac{1}{\pi} P \int_0^{1/\hat{r}} \frac{\mathrm{d}\mu_0 \Theta(\mu_0)}{\mu_0 - \mu}\right)}{\mu(Q(\mu)^2 + \pi^2)^{1/2}}. \tag{101}$$

Equations (100), (96) and (97) give

$$\psi(\mu) = \gamma(\mu)\begin{bmatrix} h(1/\nu)H(1/\xi) - h(1/\xi)H(1/\nu) \\ -k(1/\nu, 1/\xi)H(\mu) + (H(1/\nu) - H(1/\xi))h(\mu) \end{bmatrix} \tag{102}$$

with

$$\gamma(\mu) = \frac{\exp\left(-\frac{1}{\pi} P \int_0^{1/\hat{r}} \frac{d\mu_0 \Theta(\mu_0)}{\mu_0 - \mu}\right)}{\mu(Q(\mu)^2 + \pi^2)^{1/2} k(1/\nu, 1/\xi)}. \tag{103}$$

Substitution of H from (88) into (102) leads to

$$\psi(\mu) = \gamma(\mu)\left[F_0(\mu) + \int_0^{1/\hat{r}} \mathrm{d}\mu_0 \, \mathrm{e}^{-2a/\mu_0} F(\mu_0, \mu)\psi(\mu_0)\right] \tag{104}$$

with

$$F_0(\mu) = \frac{2}{\eta - 1}\left(\frac{1}{\nu}k(\mu, 1/\xi) - \frac{1}{\xi}k(\mu, 1/\nu) - \mu k(1/\nu, 1/\xi)\right) \tag{105}$$

and

$$F(\mu_0, \mu) = \frac{1}{X(-\mu_0)}\left(\frac{\nu k(\mu, 1/\xi)}{1 + \nu\mu_0} - \frac{\xi k(\mu, 1/\nu)}{1 + \xi\mu_0} - \frac{k(1/\nu, 1/\xi)}{\mu + \mu_0}\right). \tag{106}$$

From (97) and (88)

$$G_0 = \frac{\nu - \xi}{k(1/\nu, 1/\xi)}\left(\frac{2}{(\eta - 1)\nu\xi} - \int_0^{1/\hat{r}} \frac{\mathrm{d}\mu \, \psi(\mu)}{X(-\mu)(1 + \nu\mu)(1 + \xi\mu)} \mathrm{e}^{-2a/\mu}\right). \tag{107}$$

Equation (104) is a non-singular integral equation for $\psi(\mu)$. We note that F_0 is bounded on $[0, 1/\hat{r}]$ while

$$|F(\mu_0, \mu)| \leqslant \frac{K_0 \mu_0^{1/2}}{\mu + \mu_0} \tag{108}$$

for some constant K_0; γ is unbounded only as $\mu \to 0$, where $\sqrt{\mu}\gamma(\mu)$ has a finite non-zero limit. Hence $\psi(\mu) \sim K_1 \mu^{-1/2}$ as $\mu \to 0$ for some constant K_1. It follows with the use of (54) and (38) that as $x \to a$

$$M(x) \sim \frac{K_1\sqrt{\pi}}{2\sqrt{a - x}}. \tag{109}$$

Thus, we have equations which determine all of the parameters or functions appearing in equation (38) for the mass distribution $M(x)$.

5. The conditions for minimum critical mass

It was shown in II where equation (1) was originally derived, that although it describes the physical situation of minimum critical mass, there does exist a limiting value of the half-thickness a (say a_c) for which this mass is an absolute minimum. To put it otherwise, any value of $a < a_c$ gives a minimum critical mass compared with any other distribution of fuel, but the value for $a = a_c$ gives an absolute minimum. For $a > a_c$, no steady state solution exists. The form of $M(x)$ when $a = a_c$ is significantly different from the case where $a < a_c$. This was first noted in diffusion theory by Goertzel (1956), who found that it was necessary, for $a < a_c$, to add concentrated fuel at the core-reflector interface in the form of delta functions. Thus Goertzel found that the diffusion theory form of $M(x)$ was as shown in equation (37). For $a = a_c$, $\Lambda = 0$, whereas for $a < a_c$, $\Lambda > 0$.

The transport analogue of this behaviour is reflected in the behaviour of $M(x)$ near $x = a$. As we have shown in equation (109) for general values of a this diverges at $x = a$. However, there is a condition which prevents this divergence and it defines the absolute minimum half-thickness $a = a_c$ in transport theory. The appropriate condition can be found by examining the value of $M(x)$ at $x = a$, namely from equation (38)

$$M(a) = \frac{1}{\eta - 1} + G_0 \cos \lambda a + \frac{1}{2}\int_0^{1/\hat{r}} \frac{d\mu}{\mu}\psi(\mu)(1 + e^{-2a/\mu}). \tag{110}$$

Now let us insert equation (104) for $\psi(\mu)$ into equation (110) to obtain for the integral term,

$$\frac{1}{2}\int_0^{1/\hat{r}} \frac{d\mu}{\mu}(1 + e^{-2a/\mu})\gamma(\mu)\left[F_0(\mu) + \int_0^{1/\hat{r}} d\mu_0 F(\mu_0, \mu)\psi(\mu_0)\, e^{-2a/\mu_0}\right]. \tag{111}$$

The reason for the divergence is due to the behaviour of the integrand for $\mu \to 0$. This is determined by $\gamma(\mu) \sim \text{const}/\sqrt{\mu}$ because $F_0(\mu)$ and $F(\mu_0, \mu)$ both tend to constants. Thus the singularity is of order $\mu^{-3/2}$ and the integral diverges. However, we note that if we define the quantity in square brackets in (111) as $R(\mu)$, then we may write

$$R(\mu) = R(0) + \mu \tilde{R}(\mu). \tag{112}$$

If we define

$$R(0) = 0 \tag{113}$$

then the singularity in the integrand of equation (111) is removed and the integral is finite. Now $R(0) = 0$ corresponds to

$$F_0(0) + \int_0^{1/\hat{r}} d\mu_0 F(\mu_0, 0)\psi(\mu_0)\, e^{-2a/\mu_0} = 0. \tag{114}$$

Using (105) and (106) together with the definition of $k(z, \zeta)$, we find after some manipulation that a_c is given by

$$\tan \lambda(a_c - q) = \frac{\Delta_1}{\Delta_2} \tag{115}$$

where

$$\Delta_1 = \frac{2\lambda^2(\nu + \xi)}{(\eta - 1)\nu\xi} + \int_0^{1/\hat{r}} d\mu_0 \frac{\psi(\mu_0)\, e^{-2a/\mu_0}}{X(-\mu_0)} \frac{\nu\xi - \lambda^2 - \lambda^2(\nu + \xi)\mu_0}{\mu_0(1 + \nu\mu_0)(1 + \xi\mu_0)} \tag{116}$$

$$\Delta_2 = \frac{2\lambda(\lambda^2 - \nu\xi)}{(\eta - 1)\nu\xi} + \lambda \int_0^{1/\hat{r}} d\mu_0 \frac{\psi(\mu_0)\, e^{-2a/\mu_0}}{X(-\mu_0)} \frac{\nu + \xi + (\nu\xi - \lambda^2)\mu_0}{\mu_0(1 + \nu\mu_0)(1 + \xi\mu_0)}. \tag{117}$$

The parameter q arises by writing $X(i/\lambda)$ in the form

$$X(-i/\lambda) = \frac{i\lambda}{p}\, e^{i\lambda q} \tag{118}$$

where

$$p = \exp\left(\frac{\lambda^2}{\pi}\int_0^1 \frac{\Theta(\mu)\mu \mathrm{d}\mu}{1+\lambda^2\mu^2}\right) \tag{119}$$

$$q = \frac{1}{\pi}\int_0^1 \frac{\Theta(\mu) d\mu}{1+\lambda^2\mu^2} \tag{120}$$

6. Wide system approximation

It will be observed that equation (67) for $\psi(\mu)$ contains a term which is of $O(\mathrm{e}^{-2a})$. This suggests that, for systems which are more than a few mean free paths in thickness, we may neglect all such terms. If this is the case, the problem simplifies dramatically because one no longer needs to solve the integral equation (104). The solution now becomes

$$\psi(\mu) \simeq \gamma(\mu) F_0(\mu) \equiv \tilde{\psi}(\mu) \tag{121}$$

and similarly

$$G_0 \simeq \frac{2(\nu - \xi)}{(\eta - 1)\nu\xi k(1/\nu, 1/\xi)} \equiv \tilde{G}_0. \tag{122}$$

The critical equation for the minimum critical mass becomes from equation (115)

$$\tan\lambda(a_\mathrm{c} - q) = \frac{\lambda(\nu+\xi)}{\lambda^2 - \nu\xi}. \tag{123}$$

Clearly therefore the fuel mass is given by

$$M(x) = \frac{1}{\eta - 1} + \tilde{G}_0 \cos\lambda x + \int_0^{1/\hat{r}} \frac{\mathrm{d}\mu}{\mu}\tilde{\psi}(\mu)\,\mathrm{e}^{-a/\mu}\cosh(x/\mu). \tag{124}$$

It will be important to assess the accuracy of this approximation in cases of practical interest.

7. Numerical examples and discussion

In this section we will solve the integral equation (104) numerically and calculate G_0 and $M(x)$. These values will then be compared with the numerical work in II and an estimate of the error involved in the numerical method used in II made. In addition, we will assess the accuracy of the wide system approximation as developed in section 6.

To carry out a numerical solution of equation (104), a number of subsidiary functions must be put in convenient forms; in particular, $\Theta(\mu)$, $X(-\mu)$ and $\gamma(\mu)$. We note from equation (72) that

$$\Theta(\mu) = \tan^{-1}(\pi/Q(\mu)). \tag{125}$$

Now it is essential to choose the branch of $\Theta(\mu)$ which varies from $\pi/2$ to 0 as μ varies from 0 to $1/\hat{r}$. From equation (63) we know that $Q(0) = 0$ and $Q(1/\hat{r}) = \infty$; however, in some instances $Q(\mu)$can change sign. In particular, we have observed that $Q(\mu)$ starts at zero, then can become negative before increasing through zero. In this case the appropriate branch of Θ must be chosen. In general, $Q(\mu)$ remains positive and equation (125) is used. However, if $Q < 0$, then we must use

$$\Theta(\mu) = \pi - \tan^{-1}(-\pi/Q(\mu)). \tag{126}$$

To illustrate this behaviour we shall use the data for graphite and water given in table 1.

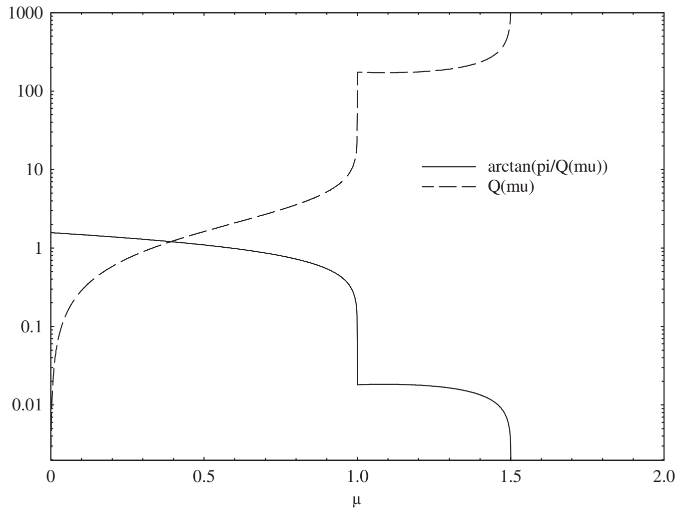

Figure 1. Functions associated with graphite.

Table 1. Moderator nuclear data in cgs units.

	Water		Graphite	
	Ackroyd	Lamarsh	Ackroyd	Lamarsh
L	2.86	2.85	49.98	59.1
L_s	5.51	5.51	20.0	19.3
$\Sigma_{\mathrm{tm}}^{\mathrm{tr}}$	1.818	2.083	0.416 7	0.397
$\Sigma_{\mathrm{fm}}^{\mathrm{tr}}$	0.306 7	0.295	0.277 8	0.328
Σ_{stm}	2.139	2.45	0.416 7	0.397
Σ_{sfm}	0.920	0.885	0.294 3	0.347
Σ_{rm}	0.035 8	0.041 9	3×10^{-3}	2.76×10^{-3}
Σ_{atm}	0.022 1	0.019 7	3.2×10^{-4}	2.4×10^{-4}
$\bar{\mu}_{\mathrm{t}}$	0.15	0.15	0.0	0.0
$\bar{\mu}_{\mathrm{f}}$	0.667	0.667	0.056	0.056
c_{sm}	0.987 99	0.990 63	0.999 23	0.999 40
c_{rm}	0.883 27	0.857 97	0.989 20	0.991 59
ν	0.188 90	0.167 02	0.047 96	0.042 56
ξ	0.093 93	0.086 25	0.119 38	0.130 75
$\hat{r}$	0.166 68	0.140 30	0.666 16	0.825 70

Figures 1 and 2 are for graphite and water, respectively and show the values of $Q(\mu)$ and $\Theta(\mu)$. The discontinuity at $\mu = 1$ is evident and in both cases $Q > 0$.

Another function of importance is $X(-\mu)$. This is defined from equation (74) by

$$X(-\mu) = -\frac{1}{\mu} \exp\left(-\frac{1}{\pi}\int_0^{1/\hat{r}} \frac{\mathrm{d}\mu_0 \Theta(\mu_0)}{\mu_0 + \mu}\right). \tag{127}$$

We know that as $\mu \to 0$, $X(-\mu) \sim \mathrm{const}\, \mu^{-1/2}$ and as $\mu \to 1/\hat{r}$, $X(-\mu)$ is finite. Thus, we use for numerical purposes the function

$$X_0(\mu) = \sqrt{\mu} X(-\mu). \tag{128}$$

Similarly, from equation (103) we know that $\gamma(\mu) \approx \mathrm{const}\, \mu^{-1/2}$, thus the function

$$\gamma_0(\mu) = \sqrt{\mu}\gamma(\mu) \tag{129}$$

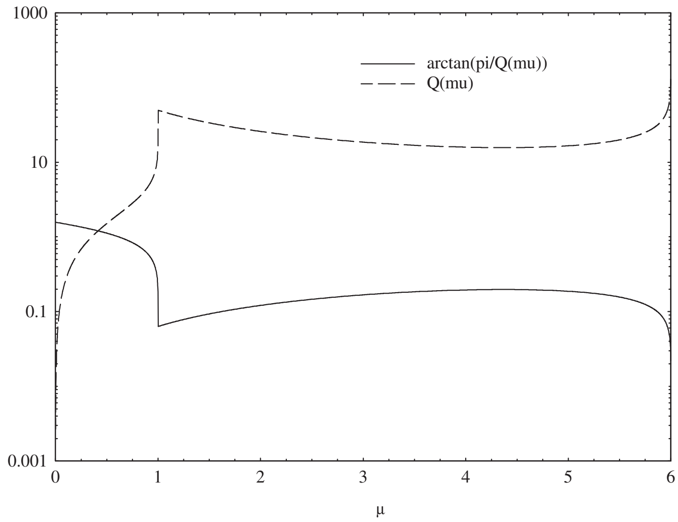

Figure 2. Functions associated with water.

is used. Both X_0 and γ_0 are numerically benign. $\gamma(\mu)$ does have a principal value integral associated with it but this is dealt with by an appropriate subroutine in the IMSL library (1998). Indeed, all supplementary quadratures and root finding (e.g., values for ν, ξ, λ) are calculated using the IMSL subroutines. We have also shown in section 4 that the solution $\psi(\mu) \sim \text{const}\, \mu^{-1/2}$ as $\mu \to 0$ and so we introduce the additional function

$$\Psi(\mu) = \sqrt{\mu}\psi(\mu). \tag{130}$$

In fact for the special case $a = a_c$, $\psi(\mu) \sim \text{const}\sqrt{\mu}$ as $\mu \to 0$, but this does not affect the usefulness of equation (130). In terms of Ψ, X_0 and γ_0 we may rewrite equation (104) as

$$\Psi(\mu) = g(\mu) + \int_0^{1/\hat{r}} \mathrm{d}\mu_0 K(\mu, \mu_0)\Psi(\mu_0)\,\mathrm{e}^{-2a/\mu_0} \tag{131}$$

where

$$g(\mu) = \frac{2}{\eta - 1}\gamma_0(\mu)\left[\frac{1}{\nu}k(\mu, 1/\xi) - \frac{1}{\xi}k(\mu, 1/\nu) - \mu k(1/\nu, 1/\xi)\right] \tag{132}$$

and

$$K(\mu, \mu_0) = \frac{\gamma_0(\mu)}{X_0(\mu_0)}\left[\frac{\nu k(\mu, 1/\xi)}{1 + \nu\mu_0} - \frac{\xi k(\mu, 1/\nu)}{1 + \xi\mu_0} - \frac{k(1/\nu, 1/\xi)}{\mu + \mu_0}\right]. \tag{133}$$

The function $k(z, \zeta)$ can be reduced to the convenient form shown below

$$k(z, \zeta) = \frac{2p(z - \zeta)}{(1 + \lambda^2 z^2)(1 + \lambda^2\zeta^2)}[(1 - \lambda^2 z\zeta)\cos\lambda(a - q) - \lambda(z + \zeta)\sin\lambda(a - q)]. \tag{134}$$

It is equation (131) which will be solved numerically. For this we use the NAG library subroutine D05ABF, details of which may be found in I. The NAG routine is particularly useful as it presents results in the form of an interpolation and so can be used directly in any subsequent quadratures to find $M(x)$, M_T and a_c. However, because of the discontinuity in slope of $Q(\mu)$ in equation (63) at $\mu = 1$, there are some corresponding discontinuities in equation (131). This leads to some oscillatory behaviour in $\Psi(\mu)$ when using the NAG library

Table 2. Fissile mass in graphite moderated system.

a (mfp)	a (cm)	M_T(N)	M_T(WS)	M_T(EX)
9.191	22.057	166.06	166.06	166.05
8.334	20	166.7	166.07	166.07
6.251	15	168.5	166.66	166.70
4.167	10	172	169.54	169.62
2.917	7	177	173.70	173.80
2.084	5	182	178.36	178.50

a (mfp): Slab half-thickness in mean free paths.
M_T(N): Numerical solution of equation (26).
M_T(WS): Wide slab approximation.
M_T(EX): Numerical solution of equation (131).

Table 3. Fissile mass in water moderated system.

a (mfp)	a (cm)	M_T(N)	M_T(WS)	M_T(EX)
16.57	9.114	31.74	–	–
16.563	9.1107	–	31.736	31.736
14.54	8	31.86	31.770	31.771
10.91	6	32.85	32.597	32.602
9.090	5	34.40	34.042	34.049
7.272	4	37.79	37.305	37.316
5.454	3	46.31	45.600	45.621

routine, even with 100 terms in the Chebyshev series. For this reason, we have developed another method of solving (131) based upon iteration. That is we write

$$\Psi_{n+1}(\mu) = g(\mu) + \int_0^{1/\hat{r}} d\mu_0 K(\mu, \mu_0)\Psi_n(\mu_0)\, e^{-2a/\mu_0} \tag{135}$$

with $\Psi_0(\mu) = g(\mu)$. This procedure converges rapidly and after three iterations little difference is noted between the solutions. In evaluating this iterative process we stored the Ψ_n values by means of an interpolation formula for use in the $n+1$ iteration. Ψ itself has no physical meaning and therefore only integrals over it are required. Any residual oscillations around $\mu = 1$ are therefore averaged out and we are confident that our numerical results are accurate to the figures given.

We illustrate the theory by means of the moderator data given in table 1 using Ackroyd's set. For graphite, we have calculated the mass distribution function and the total mass content for a range of slab half-thicknesses. The exact results are compared with those from the wide slab approximation and some results obtained by solving equation (26) directly (II). Table 2 gives the results for graphite.

The first row in table 2 corresponds to the critical value a_c and the units are in cm. M_T is defined by equation (65). The values of a_c for the wide lab and exact cases are identical to six significant figures. The values of M_T are also correct to four significant figures. We note that the numerical method of II becomes progressively less accurate as the thickness decreases. This is not unexpected as the numerical method employed has difficulty in reproducing the very steep rise in $M(x)$ near the boundaries. We also note that the wide slab approximation is very accurate. The error will increase for a much smaller than a few mean free paths but such sizes are in a region where the equation itself is no longer valid.

Analogous results for water are given in table 3 and conclusions similar to those for graphite can be drawn.

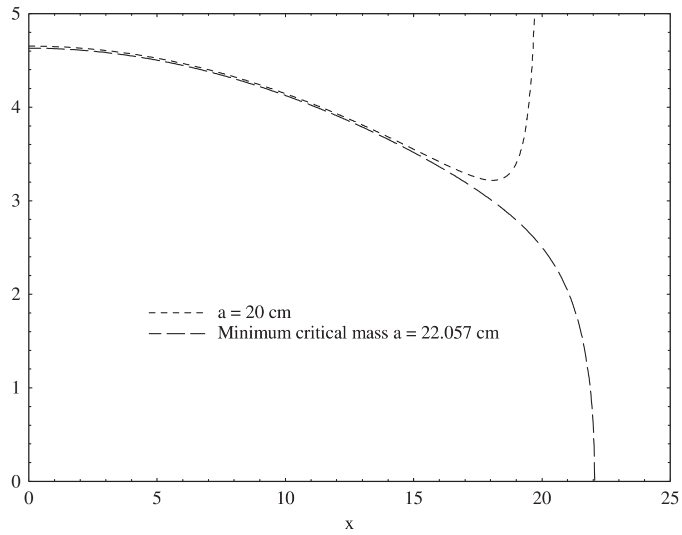

Figure 3. Mass distribution function for minimum critical mass case and for $a = 20$ cm.

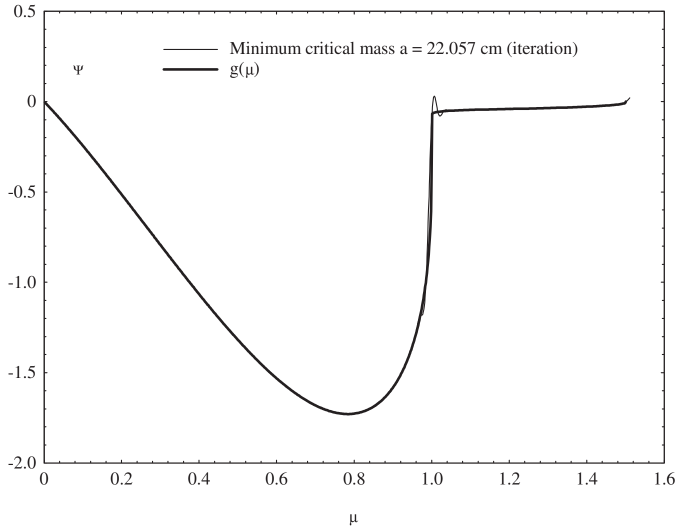

Figure 4. Solution of integral equation for graphite minimum critical mass case, $a = a_c$.

We illustrate in figure 3 the form taken by $M(x)$ by considering the special case of $a = a_c$ and the case $a = 20$ cm for graphite. The difference is evident, with the value of $M(a_c)$ being zero whilst that for $a = 20$ cm is infinite. It is also of interest to observe how the solution $\Psi(\mu)$ varies for the two cases. Figures 4 and 5 show this behaviour. Figure 4 is for $a = a_c$ and figure 5 for $a = 20$ cm. We note that for $a = a_c$, $\Psi < 0$ for all values of $\mu(0, 1/\hat{r})$, whereas for $a = 20$ only a small region of $\Psi(\mu)$ is negative. It is this dramatic change in behaviour that is responsible for the differences in $M(x)$. We also note in figures 4 and 5 the oscillations around $\mu = 1$. This is perhaps an example of the Gibbs phenomenon since $g(\mu)$ has a discontinuity at $\mu = 1$. Analogous behaviour is observed for water.

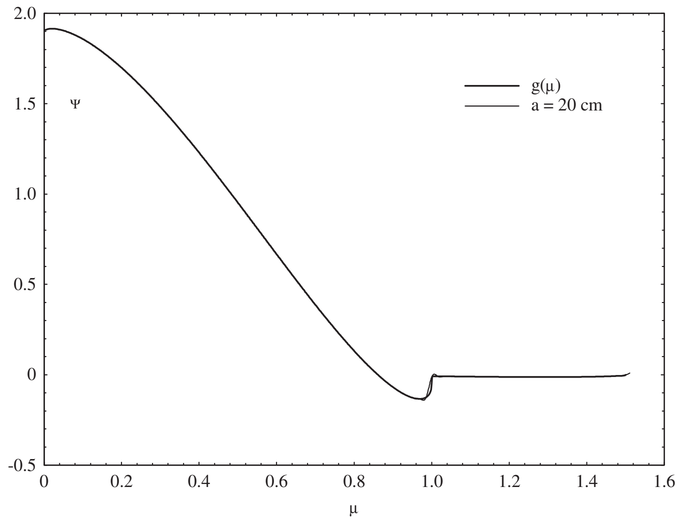

Figure 5. Solution of integral equation for graphite, $a = 20$ cm.

8. Conclusions

The exact solution derived in this paper forms a benchmark against which to assess the accuracy of less accurate but more convenient schemes of solution. We have noted in particular that the wide slab approximation, which arises when terms of $O(\mathrm{e}^{-2a})$ are neglected, is exceptionally accurate and can be used with confidence in calculating $M(x)$, a_{c} and M_{T}. The numerical method used in II, however, is restricted due to the difficulty it has in dealing with the steep rise in $M(x)$ near $x = a$. However, in the case of $a = a_{\mathrm{c}}$ it is very accurate.

Appendix. Some mathematical relationships

Several identities are used in the text; here we outline their proof.

Relationship 1

The diffusion kernel $P_{\mathrm{t}}(x)$ may be written as

$$P_{\mathrm{t}}(x) = A_{\mathrm{t}}\,\mathrm{e}^{-\nu x} + \frac{1}{2}\int_0^1 \frac{\mathrm{d}\mu}{\mu} g(c_{\mathrm{sm}}, \mu)\,\mathrm{e}^{-\Sigma_{\mathrm{tm}} x/\mu} \tag{A1}$$

where

$$A_{\mathrm{t}} = \frac{\nu(1-\nu^2)}{c_{\mathrm{sm}}(\nu^2 - 1 + c_{\mathrm{sm}})}. \tag{A2}$$

The Fourier transform of $P_{\mathrm{t}}(x)$ is

$$\bar{P}_{\mathrm{t}}(k) = \frac{\tan^{-1} k}{k - c_{\mathrm{sm}} \tan^{-1} k}. \tag{A3}$$

The integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \cos(\lambda x) P_{\mathrm{t}}(|x|) = \frac{2\nu A_{\mathrm{t}}}{\nu^2 + \lambda^2} + \int_0^1 \frac{\mathrm{d}\mu\, g(c_{\mathrm{sm}}, \mu)}{1 + \lambda^2\mu^2}. \tag{A4}$$

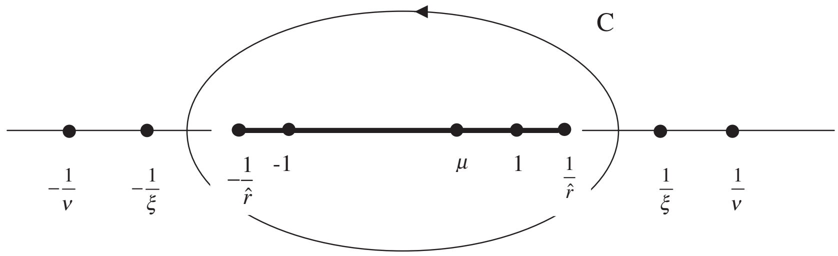

Figure 6. Singularities in the z-frame.

But

$$\int_{-\infty}^{\infty} \mathrm{d}x \cos(\lambda x) P_{\mathrm{t}}(|x|) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x (\mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{e}^{-\mathrm{i}\lambda x}) P_{\mathrm{t}}(|x|) = \frac{1}{2} (\bar{P}_{\mathrm{t}}(\lambda) + \bar{P}_{\mathrm{t}}(-\lambda)) \tag{A5}$$

and using the fact that $\bar{P}_{\mathrm{t}}(k) = \bar{P}_{\mathrm{t}}(-k)$, we find

$$\frac{2\nu A_{\mathrm{t}}}{\lambda^2 + \nu^2} + \int_0^1 \frac{\mathrm{d}\mu g(c_{\mathrm{sm}}, \mu)}{1 + \lambda^2 \mu^2} = \frac{\tan^{-1} \lambda}{\lambda - c_{\mathrm{sm}} \tan^{-1} \lambda} \tag{A6}$$

as used in equation (48).

Relationship 2

We need a simple expression for the integral

$$\int_{-\infty}^{\infty} \mathrm{d}x \cos(\lambda x) P_{\mathrm{tf}}(|x|) = \frac{2\nu C_{\mathrm{t}}}{\nu^2 + \lambda^2} + \frac{2\xi C_{\mathrm{f}}}{\xi^2 + \lambda^2} + \frac{1}{2} \int_0^1 \frac{\mathrm{d}\mu \mu B_0(\mu)}{1 + \lambda^2 \mu^2} + \frac{1}{2} \int_1^{1/\hat{r}} \frac{\mathrm{d}\mu \mu B_1(\mu)}{1 + \lambda^2 \mu^2}. \tag{A7}$$

By using a relation analogous to (A5), we find

$$\int_{-\infty}^{\infty} \mathrm{d}x \cos(\lambda x) P_{\mathrm{tf}}(|x|) = \bar{P}_{\mathrm{tf}}(k)$$

where

$$\bar{P}_{\mathrm{tf}}(\lambda) = \frac{\tan^{-1}(\lambda/\hat{r})}{\lambda - c_{\mathrm{rm}} \hat{r} \tan^{-1}(\lambda/\hat{r})} \frac{\tan^{-1} \lambda}{\lambda - c_{\mathrm{sm}} \tan^{-1} \lambda} \tag{A8}$$

hence we have (49) in the text.

Relationship 3

We now have to prove equations (59) and (60). Consider the integral

$$I = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}z z \log((z+1)/(z-1)) \log((\hat{r}z+1)/(\hat{r}z-1))}{(z-\mu) M_{\mathrm{t}}(z) M_{\mathrm{f}}(z)} \tag{A9}$$

where

$$M_{\mathrm{t}}(z) = 1 - \frac{1}{2} c_{\mathrm{sm}} z \log\left(\frac{z+1}{z-1}\right) \tag{A10}$$

$$M_{\mathrm{f}}(z) = 1 - \frac{1}{2} c_{\mathrm{rm}} z \hat{r} \log\left(\frac{\hat{r}z+1}{\hat{r}z-1}\right). \tag{A11}$$

We now take the contour C as shown in figure 6. It is assumed in all cases that $\hat{r} < 1$.

By expanding the contour outwards, we pick up the poles at $\pm 1/\nu$ and $\pm 1/\xi$, hence

$$I = -\frac{8\nu\mu C_{\mathrm{t}}}{1-\nu^2\mu^2} - \frac{8\xi\mu C_{\mathrm{f}}}{1-\xi^2\mu^2}. \tag{A12}$$

Contracting the contour onto the cut and noting the embedded pole at $z = \mu$, we find

$$\begin{aligned} I = 2\mu P.\int_0^1 \frac{\mathrm{d}\mu'\mu' B_0(\mu')}{\mu'^2-\mu^2} + 2\mu\int_1^{1/\hat{r}} \frac{\mathrm{d}\mu'\mu' B_1(\mu')}{\mu'^2-\mu^2} + \mu g(c_{\mathrm{sm}},\mu) g(c_{\mathrm{rm}},\hat{r}\mu) \\ \times \left[-\pi^2 + \left(-L(\mu) + \frac{1}{2}c_{\mathrm{sm}}\mu(\pi^2 + L^2(\mu))\right)\right. \\ \left.\times\left(-L(\hat{r}\mu) + \frac{1}{2}c_{\mathrm{rm}}\hat{r}\mu(\pi^2 + L^2(\hat{r}\mu))\right)\right] \qquad 0 \leqslant \mu \leqslant 1 \end{aligned} \tag{A13}$$

and for $1 \leqslant \mu \leqslant 1/\hat{r}$

$$\begin{aligned} I = 2\mu\int_0^1 \frac{\mathrm{d}\mu'\mu' B_0(\mu')}{\mu'^2-\mu^2} + 2\mu P.\int_1^{1/\hat{r}} \frac{\mathrm{d}\mu'\mu' B_1(\mu')}{\mu'^2-\mu^2} + \frac{\mu\bar{L}(\mu)}{1-\frac{1}{2}c_{\mathrm{sm}}\mu\bar{L}(\mu)} g(c_{\mathrm{rm}},\hat{r}\mu) \\ \times\left(L(\hat{r}\mu) - \frac{1}{2}c_{\mathrm{rm}}\hat{r}\mu(\pi^2 + L^2(\hat{r}\mu))\right) \end{aligned}$$

where $\bar{L}(\mu) = \log((\mu+1)/(\mu-1))$.

Equating (A12) and (A13) gives (59) and (60). Also we note that although we have assumed that $\nu < \hat{r}$, the same result arises if $\nu > \hat{r}$. In that case the poles at $z = \pm 1/\nu$ occur embedded in the cut in the regions $(-1/\hat{r}, -1)$, $(1, 1/\hat{r})$.

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